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Pair Production by a Constant External Field in Noncommutative QED

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abstract

In this paper we study QED on the noncommutative space in the constant electro-magnetic field background. Using the explicit solutions of the noncommutative version of Dirac equation in such background, we show that there are well-defined in and out -going asymptotic states and also there is a causal Green's function. We calculate the pair production rate in this case. We show that at tree level noncommutativity will not change the pair production and the threshold electric field. We also calculate the pair production rate considering the first loop corrections. In this case we show that the threshold electric field is decreased by the noncommutativity effects.

1 Introduction

Noncommutative spaces naturally come about when one studies D-brane worldvolume theory in the B-field background [1]. Such noncommutative spaces can be characterized by the coordinate operators, $\hat{X}^{\mu}$, satisfying

$$[\hat{X}_{\mu}, \hat{X}_{\nu}] = i\theta_{\mu\nu}, \tag{1.1}$$

where $\theta_{\mu\nu}$ is the noncommutative parameter and is of dimension of (length)^2 . The field theory formulated on these spaces, the noncommutative field theory, is then described by field operators which are functions of $\hat{X}$. However, using the Weyl-Moyal correspondence [2, 3, 4] one can show that instead of operator valued functions, it is enough to take the corresponding commutative theory and replace the product of fields by the *star* product:

$$(f * g)(x) = exp(\frac{i}{2}\theta_{\mu\nu}\partial_{x_{\mu}}\partial_{y_{\nu}})f(x)g(y)|_{x=y}, \tag{1.2}$$

where f and g are two arbitrary functions and assumed to be infinitely differentiable. Then, in order to quantize the theory one should specify the Hilbert space (or equivalently the measure, in the path integral formulation). Thanks to the properties of star product under the integral over the space-time, star product in the quadratic terms of the action can be removed, suggesting that the asymptotic states of free field theory can be consistently chosen the same as the corresponding commutative theory; i.e. perturbative Hilbert space of a noncommutative field theory is that of the commutative one [3]. However, this argument is true if, the conjugate momentum of the field is the same as its commutative counter-part, for which the definition of star product (1.2) implies that the time components of $\theta_{\mu\nu}$, θ_{0i}, should be zero and hence, we will restrict ourselves only to these cases. In fact it has been shown that theories with time-like noncommutativity ($\theta_{0i} \neq 0$, $\theta_{ij} = 0$) are not unitary [5], however those with light-like noncommutativity lead to well-defined quantum theories [6].

Here, we investigate some more possible phenomenological aspects of noncommutative field theories, and in particular noncommutative version of QED. Some other phenomenological consequences of noncommutative standard model have been addressed in [7]. So, first we start with introduction of pure noncommutative gauge theories, then we add fermions and as usual, form of the interaction terms are fixed using the noncommutative gauge invariance.

Performing explicit calculations of two and three point functions, it has been shown that NCQED satisfies on-shell Ward identity and is renormalizable (at one loop) [4, 8, 9]. Studying effective interaction vertex it has been shown that [4]:

i) magnetic dipole moment of a noncommutative Dirac particle at one loop level has a *spin independent* part which is directly proportional to noncommutativity parameter, θ;

ii) any moving noncommutative Dirac particle shows *electric dipole* effects, which of course receives quantum corrections at one loop.

The behaviour of NCQED under discrete symmetries (C, P and T) have also been addressed and shown that for space-like noncommutativity ($\theta_{0i} = 0$) the theory is parity preserving while CP violating. More precisely, under charge conjugation NCQED on R^4_θ is mapped into a NCQED on $R^4_{-\theta}$. Also it has been shown that the theory preserves CPT [10]. Hence it is plausible to look for some generalization of the usual CPT theorem which is also true for the Lorentz non-invariant cases like the noncommutative Moyal plane [18].

In this paper we study pair production in NCQED by a constant electro-magnetic field background. The constant background has been previously considered in [11]. The peculiar feature of the constant electro-magnetic field is the appearance of gauge non-invariant quantities. This is due to the fact that the "trace" which makes the operators to be gauge invariant, in the noncommutative case is replaced by the integration over the space-time, and besides the integration one should also neglect the surface terms at infinity, which of course is not true for the constant field strength case [11].

To find the pair production rate, we use Nikishov method which is based on explicit solutions of the corresponding Dirac equation [12, 13] (for a more recent review and more detailed references see also [14]).

Solving the noncommutative Dirac equation in the constant electro-magnetic field background we show that, similar to the commutative case, there are well-defined in and out going fermionic states and hence, there is also a well-behaved Green's function. Having the proper states and propagator we work out the rate of pair production in the unit volume, unit time. In the tree level, we show that the noncommutative effects do not appear in the physical rate. As a result, at tree level, the threshold electric field for the pair production is not affected by noncommutative corrections.

In addition we perform the calculations considering the first loop effects on the magnetic dipole moment of electron in NCQED [4]. In this case we show that the threshold electric field receives some corrections due to noncommutativity.

2 NCQED, the action

In order to get the noncommutative Dirac equation coupled to NCU(1) theory, we first build the action:

i)Pure Gauge theory

The action for the pure gauge theory is

$$S = \frac{1}{4\pi}\int F_{\mu\nu} * F^{\mu\nu} d^4x = \frac{1}{4\pi}\int F_{\mu\nu}F^{\mu\nu} d^4x \ , \tag{2.1}$$

with

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + ie\{A_\mu, A_\nu\}_{MB} \ . \tag{2.2}$$

In the above e is the gauge coupling constant and the MB stands for Moyal bracket defined as $\{f, g\}_{MB} = f*g - g*f$. One can show that the above action enjoys the noncommutative gauge transformations[1, 8]

$$\begin{aligned} A_\mu \to A'_\mu = U(x) * A_\mu * U^{-1}(x) + \frac{i}{g}U(x) * \partial_\mu U^{-1}(x) \ , \\ U(x) = exp*(i\lambda), \qquad U^{-1}(x) = exp*(-i\lambda) \ , \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} exp*(i\lambda(x)) \equiv 1 + i\lambda - \tfrac{1}{2}\lambda*\lambda - \tfrac{i}{3!}\lambda*\lambda*\lambda + ... \ , \\ U(x) * U^{-1}(x) = 1 \ . \end{aligned} \tag{2.4}$$

Since here we are only interested in NCQED, we choose λ, A_μ to be in $U(1)$ algebra. However, this can easily be extended to $U(n)$ valued functions giving rise to NCU(n) theory.

ii)Fermionic Part

Fermions can be added to the above gauge theory, developing the definition of "covariant derivative". In the NCQED, it has been shown that there are two different kinds of covariant derivatives related by charge conjugation. In other words, there are two different types of fermions which are mapped into each other by charge conjugation, hence they can be called positively or negatively charged fermions [8, 10]. The explicit form of the covariant derivative for the positively charged particles is

$$D^+_\mu\psi(x) \equiv \partial_\mu\psi(x) - ie(A_\mu * \psi)(x) \ , \tag{2.5}$$

while for the particles with the negative charge it is

$$D^-_\mu\psi(x) \equiv \partial^\mu\psi(x) + ie(\psi * A_\mu)(x) \ .$$

In this paper we only consider the D^+ case, and the other, D^- can be recovered by just sending θ to $-\theta$. The fermionic part of NCQED action is then

$$S_f = \int d^4x \bar{\psi} * (-i\gamma^\mu D^+_\mu - m)\psi \ . \tag{2.6}$$

It is easy to verify that this action is also invariant under NCU(1) transformation defined by $\psi \to U * \psi$ and (2.3). Using the definition of the covariant derivative one can verify that:

$$\{D^\pm_\mu, D^\pm_\nu\}_{MB} = \mp ieF_{\mu\nu} \ . \tag{2.7}$$

3 Pair production amplitude

To find the pair production rate, we solve the noncommutative Dirac equation,

$$(-i\gamma^\mu\partial_\mu - m)\psi + \ e\gamma^\mu A_\mu * \psi = 0 \ , \tag{3.1}$$

in the constant field background. Since there is a Lorentz transformation which maps constant electric and magnetic field ($\vec{E}$ and $\vec{B}$) into parallel $\vec{E}$ and $\vec{B}$ *, we only consider the parallel electric and magnetic fields here. We choose our x^3 axis to be along $\vec{E}$. Then the corresponding A_μ field can be taken as

$$A_\mu = (0,\ 0,\ Bx_1,\ -Et) \ , \tag{3.2}$$

where t is the time coordinate (x^0). Inserting this A_μ into the Dirac equation we obtain

$$(\gamma^\mu \Pi_\mu - m)\psi = 0 \ ; \tag{3.3}$$

$$\begin{cases} \Pi_\mu = -i\partial_\mu + eA_\mu \qquad \mu \neq 2 \ , \\ \Pi_2 = -i\partial_2 + eBx_1 + \frac{i}{2}eB\ \theta_{1j}\partial_j \ . \end{cases} \tag{3.4}$$

Following the lines of [12, 13], it is more convenient to use the squared Dirac equation (for our conventions see the Appendix)

$$(\Pi^2 + S - m^2)\Psi = 0 \ , \tag{3.5}$$

*However, we note that this is not true for the cases in which both Lorentz-invariants, namely $E^2 - B^2$ and $E \cdot B$, are zero. Hence our arguments is not covering those cases.

where

$$S = -\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \ , \tag{3.6}$$

and with our choice the non-zero components of F are $F_{30} = -F_{03} = E, \ F_{12} = -F_{21} = B$. Then the fermion field, ψ, can be obtained as

$$\psi = (m - \Pi_\mu\gamma^\mu)\Psi \ . \tag{3.7}$$

In order to solve (3.5), first we find the eigenvectors and eigenvalues of matrix S:

$$S\Gamma_i = s_i\Gamma_i \ , \qquad i = 1, 2, 3, 4 \ . \tag{3.8}$$

For our purpose it is enough to consider only two of these, e.g. $i = 1, 2$, which hereafter we will denote them by $+$ and $-$:

$$s_\pm = \pm eB - ieE \ , \tag{3.9}$$

and

$$\Gamma_+ = (1,\ 0,\ 1,\ 0) \ , \qquad \Gamma_- = (0,\ 1,\ 0,\ -1) \ .$$

Then, Ψ can be decomposed into the matrix part which is proportional to $\Gamma_\pm$ and the functional part

$$\Psi_\pm = Z_\pm(x)\ \Gamma_\pm \ , \tag{3.10}$$

where $Z_\pm$ satisfy the equations

$$(\Pi^2 + s_\pm - m^2)Z_\pm = 0 \ . \tag{3.11}$$

Noting that $x^2,\ x^3$ do not appear in the equation (3.11), we use the following ansatz for $Z_\pm$,

$$Z_\pm = N\ exp(ip_2x_2 + ip_3x_3)\ F_\pm(x_1, t) \ . \tag{3.12}$$

Plugging this ansatz into (3.11), defining

$$P_2 = p_2 - \frac{1}{2}eB\ \theta_{1j}p_j \ , \tag{3.13}$$

$$eE\tau^2 = (p_3 - eEt)^2 \ , \tag{3.14}$$

$$eB\rho^2 = (P_2 + eBx_1)^2 \ , \tag{3.15}$$

and separating the variables, i.e.

$$F_\pm(\rho, \tau) = \chi_\pm(\rho)\ \Phi_\pm(\tau) \ ,$$

we get

$$(\frac{\partial^2}{\partial\rho^2} - \rho^2 \pm 1)\chi_\pm = K_\pm \chi_\pm \ , \tag{3.16}$$

$$(\frac{\partial^2}{\partial\tau^2} + \tau^2 + i + \lambda_\pm)\Phi_\pm = 0 \ , \tag{3.17}$$

where $K_\pm$ are constants and

$$\lambda_\pm = \frac{m^2 - eBK_\pm}{eE} \ . \tag{3.18}$$

Eq. (3.16) is basically the Schroedinger equation for a harmonic oscillator, hence:

$$K_+ = 2(n+1) \quad , \quad K_- = 2n \ , \quad n = 0, 1, 2, ... \ , \tag{3.19}$$

and $\chi_\pm$ can be written in terms of Hermite polynomials: $\chi_\pm = e^{-\frac{1}{2}\rho^2} \ H_n(\rho)$. Solutions of eq. (3.17) are the parabolic-cylinder functions [15]

$$\begin{cases} \Phi_{1+} = D_\nu(-(1-i)\tau) \\ \\ \Phi_{2+} = D_{\nu*}((1+i)\tau) \end{cases} ; \begin{cases} \Phi_{1-} = D_{\nu*}(-(1+i)\tau) \\ \\ \Phi_{2-} = D_\nu((1-i)\tau) \end{cases} \ , \tag{3.20}$$

where

$$\nu = \frac{1}{2}(1 + i\lambda_\pm) \ . \tag{3.21}$$

Not all the above mentioned solutions, (3.20), are linearly independent; Φ_{i+} and Φ_{i-} form two complete sets. ¿From the asymptotic expansions of D_ν functions we observe that our $\Phi_{i\pm}$ are leading to some well-defined states as $t \to \pm\infty$. Actually, Φ_{i+} are those with positive frequency solutions and Φ_{i-} with negative frequency, as $t \to \infty$. These solutions are related through

$$\begin{cases} \Phi_{1+} = \beta\Phi_{2-} + \zeta\Phi_{2+} \\ \\ \Phi_{2+} = \beta^*\Phi_{1-} + \zeta^*\Phi_{1+} \ , \end{cases} \tag{3.22}$$

with $\beta = e^{i\pi\nu}$, and $-|\beta|^2 + |\zeta|^2 = 1$. Now that we have the final fermionic solution, the absolute probability for pair production is then, $|\beta|^2 = e^{-\pi\lambda}$. As it is seen the effects of noncommutativity has been totally disappeared in the final result but, still one should work out the pair production rate per unit volume, unit time. The family of our fermionic solutions is characterized by quantum numbers $P_2, p_3, \ n$ and also $\pm$ sings, corresponding to two spin states. Hence to find the full rate one should sum over all these set of quantum numbers. In

order to normalize our states (and also regularize our calculations) let us put our system in a box of sides L, then the average number of pairs produced is

$$\bar{\mathcal{N}} = \int dP_2 dp_3 \sum_n (|\beta_+|^2 + |\beta_-|^2) \, \frac{L^2}{(2\pi)^2} \ . \tag{3.23}$$

Since there is no explicit P_2, p_3 dependence in β's, using eqs. (3.14), (3.15), integration over p_3 and P_2 can be replaced by [12, 14]

$$\int dp_3 \longrightarrow eET \quad , \quad \int dP_2 \longrightarrow eBL \ ,$$

So altogether the pair production rate, $\frac{\bar{\mathcal{N}}}{L^3 T}$, is

$$I_0(E,B) = \frac{\alpha EB}{\pi} \ exp(-\frac{\pi m^2}{eE}) \ \coth(\frac{\pi B}{E}) \ , \tag{3.24}$$

which is exactly the same as the commutative results of [13, 14].

Here we should remind that in all the above manipulations, instead of the eigen-values of $\frac{\partial}{\partial x_2}$ operator, p_2, we have used the P_2, which is the eigen-value for the "physical" momenta along x_2 when we have a non-zero θ parameter. This "physical" quantum number is actually counter-part of the x'-coordinate system introduced in [11] and is not invariant under the NC$U(1)$ gauge transformations. Then, as explained in [11], the extra factor arising in going from p_2 to P_2 frame can be absorbed in the normalization factor, N, in Eq.(3.12). Although we have not presented here the result (3.24) is invariant under the NC$U(1)$ gauge transformations defined by (2.3) and (2.5), and E and B are the electric and magnetic components of (2.2).

All of the above calculations are done in the classical level. The quantum (loop) effects can also be included if we consider the anomalous magnetic moment of Dirac particles [16, 17]. In the usual QED, this can easily be done by replacing $s_\pm$, (3.9), with $\pm\frac{g}{2}eB - ieE$, where g is the gyro-magnetic ratio. However, for the NCQED case as shown in [4], at one loop level Dirac particle will also show a spin independent magnetic moment. Taking this into account, the proper $s_\pm$ for the NCQED at one loop level is

$$s_\pm = \pm\frac{g}{2}eB - ieE + \ \frac{e\alpha\gamma_E}{3\pi} m^2 \vec{\theta}.\vec{B} \ , \tag{3.25}$$

where

$$g - 2 = \frac{\alpha}{\pi} \quad \text{and} \quad \theta_i \equiv \epsilon_{ijk}\theta_{jk} \ , \tag{3.26}$$

and $\gamma_E = \gamma_{\rm Euller}$. Inserting these values for $s_\pm$ and repeating all the computations, the pair production rate up to first loop is

$$I_{\rm 1st \ loop} = \frac{\alpha EB}{\pi} \ exp[-\frac{\pi m^2}{eE}(1 - \frac{e\alpha\gamma_{\rm E}}{3\pi}\vec{\theta}.\vec{B})] \times \frac{\cosh(\frac{\pi g B}{2E})}{\sinh(\frac{\pi B}{E})} \ . \tag{3.27}$$

The interesting point is that, the threshold electric field is reduced by the noncommutativity effects. This possible change in the threshold electric field due to noncommutativity can have important astrophysical and cosmological consequences, where we have a very strong magnetic field, e.g. for the neutron stars. This change in the pair production rate can be used to put some (lower) bound on θ.

For the pair annihilation without photon emission amplitude, since our theory is T violating, it is not the same as pair production rate. However, using the CPT invariance of the theory [10], this amplitude is related to that of the pair production by $\theta \to -\theta$ transformation.

The other comment we should make is that, here we only present the calculations for spin one-half particles. However, all of our discussions through the lines of [14] can be generalized to particles with arbitrary spin. As it is expected, at classical level, we obtain the commutative results, however at the one loop we expect to see the noncommutative effects. These effects are presumably the same as (3.27) but now the factor $\frac{1}{3\pi}$ in the term proportional to $\theta.\vec{B}$ is replaced by the proper numeric factor.

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Appendix : The γ matrix conventions

In this paper we used the following conventions:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} ,$$

which lead to

$$\eta_{\mu\nu} = \mathrm{diag}(+1,\ -1,\ -1,\ -1) \ .$$

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